Use of Negative Weights in Linear Quadratic Regulator Synthesis

H. Ohta* and M. Kakinuma†
Nagoya University, Chikusa-ku, Nagoya 464, Japan
and
P. N. Nikiforuk‡
University of Saskatchewan, Saskatoon, Saskatchewan S7N OWO, Canada

An inverse problem of optimal linear quadratic regulators (LQR) is examined for single-input systems, and the selection of the weighting matrices that achieve a specified pole location is also discussed. In particular, the Kessler polynomial is used as a desirable pole location, and the weighting matrices are derived in an analytical form. Although this pole specification results in the use of some negative diagonal weights in the performance index, the existence and uniqueness of the Riccati solution are guaranteed by Molinari's theorem. Through the sacrifice of the circle condition, it is shown that some of the deficiencies of the LQR controllers are avoided, and that several characteristics (which classical controllers provide, but which modern methods cannot) are retained. An application to roll autopilot systems for missiles is given to illustrate and substantiate the proposed method, as well as to compare it with the conventional LQR.

Introduction

To date, optimal linear quadratic regulators (LQR) have been discussed under the assumption that the state diagonal weights are not negative. As a result, the circle condition is satisfied, and good crossover characteristics are guaranteed.[1] Because of these desirable characteristics, many works have been reported in which the LQR method is used for pole assignment.[2] On the other hand, the damping characteristics may not be satisfactory for systems of high relative degree because the closed-loop poles are asymptotically of a Butterworth configuration. The resulting controlled system, based upon conventionally selected weighting coefficients, will usually have a large bandwidth. Further, the system may become unstable if high-frequency dynamics have been neglected for purposes of controller design. The gain rolloff property is always -20 dB/dec, so that attenuation of high-frequency noise is small.

Some research has also been reported on the LQR methods with negative state weights. Hayase[3] derived the weighting conditions for constructing stable time-invariant systems. Shin and Chen[4] examined proportional-intergal-derivative (PID) compensators for a second-order system, and showed that negative diagonal weights may appear when the controllers are designed, using the Cohen-Coon method or the Ziegler-Nichols method. They stated that the positive definite condition of the state weights may not be essential in practice. Since high-frequency unmodeled modes are always present in a real plant, an LQR controller may become unstable even if the mathematical model satisfies the circle condition and has sufficient gain and phase margin.[5] It is possible, therefore, to improve the LQR method if negative diagonal state weights are used in the performance index. Since the state variables used in the performance index are connected with a state equation, the variables cannot be independent of each other. Thus, a possibility remains that the weights may be selected more freely. In general, however, there is no guideline for selecting negative weights.

In this paper an inverse problem[6] of the LQR controller is examined for single-input systems, and the weighting matrices that achieve a specified closed-loop pole location are explicitly derived. In particular, it shown that the deficiencies of the LQR method are decreased when the Kessler pole location is used as a desirable pole specification. A problem associated with this pole specification is that it leads to the use of some negative diagonal weights in the performance index, and that the circle condition can no longer hold. Sacrificing this condition, it is shown that several characteristics, provided by classical controllers but not by modern methods, are retained.

An application to roll autopilot systems for missiles is given to illustrate and substantiate the method. The stability robustness associated with unmodeled dynamics is also examined, using singular values.

Design Procedure

Consider a single-input linear time-invariant system

$$\dot{x} = Ax + Bu \tag{1}$$

where x is an n-dimensional state vector and A and B are constant matrices of appropriate dimension. Assume that the pair (A, B) is of the controllable canonical form

$$A = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2}$$

Let the performance index of the system be given by

$$J = \int_0^\infty (x^TQx + ru^2)\, \mathrm{d}t \tag{3}$$

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*Assistant Professor, Department of Aeronautical Engineering. Member AIAA.
†Graduate Student, Graduate School of Engineering.
‡Professor, School of Engineering.

where it is assumed that r is a positive scalar and Q is a diagonal matrix

$$Q = \mathrm{diag}[q_1, q_2, \dots, q_n] \tag{4}$$

Control Law

The optimal control u that minimizes Eq. (3) satisfies the relation

$$\phi_c(s)\phi_c(-s) = \phi(s)\phi(-s)[1 + (1/r)\phi_0^T(-s)Q\phi_0(s)] \tag{5}$$

$$\phi_c(s) = \left[\sum_{i=0}^{n} d_i\,(s/\omega_c)^{n-i}\right](\omega_c)^n$$

$$\phi(s) = \det[sI - A] = \sum_{i=0}^{n} a_i\, s^{n-i}$$

$$\phi_0(s) = (sI - A)^{-1} B$$

where $\phi_c(s)$ is a closed-loop characteristic polynomial, ω_c is a closed-loop crossover frequency, $a_0 = d_0 = 1$, and d_i are the coefficients of a desirable characteristic polynomial determined by the design requirements. The solution of Eq. (5) is of the state feedback form

$$u = -Kx, \qquad K = [k_1, k_2, \dots, k_n] \tag{6}$$

and leads to the block diagram shown in Fig. 1, where y represents x_1. One of the purposes of this paper is to determine the weighting matrix Q, which achieves a desirable pole location given by $\phi_c(s)$ in Eq. (5). The Kessler type of pole location is used for this purpose.

Kessler Polynomial Form

When the weight r is decreased or the matrix Q is increased, some of the poles of the LQR with a positive semidefinite matrix Q approach equivalent zeros, and the remainder approach infinity with the Butterworth configuration. This shows that, for a system of high relative degree, the poles going to infinity are placed close to the imaginary axis. When the Kessler polynomial form is used as a pole specification, the closed-loop characteristic polynomial $\phi_c(s)$ is given by

$$\phi_c(s) = \left[\sum_{i=0}^{n} 2^{-i(i-1)/2}(s/\omega_c)^{n-i}\right](\omega_c)^n \tag{7}$$

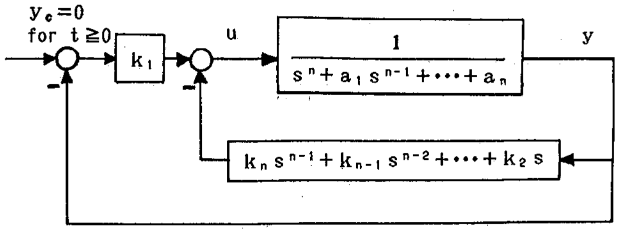

Fig. 1 The *n*th-order controlled plant.

These poles are distributed inside a fan whose angle from the negative real axis is less than 60 deg, and, therefore, good damping characteristics are expected. The Kessler pole locations of order 1–5 are shown in Table 1. It is recognized in the table that the damping characteristics of the fifth-order Kessler configuration are better than those of the Butterworth configuration, because the maximum angle from the negative real axis is 70 deg in the Butterworth configuration and only 49.5 deg in the Kessler configuration.

Solution of the Inverse Problem

Choice of Quadratic Weights

The first aim of this paper is to provide a general solution of the pole assignment problem in Eq. (5). A relationship between the selected weighting matrix and the existence of the solution of the algebraic Riccati equation (ARE) will be discussed later.

Lemma 1. Consider the LQR problem given in Eqs. (1–4). The weighting coefficients providing the pole location of Eq. (5) are given by

$$q_{n-m+1}/r = \sum_{i=k}^{\ell} [(-1)^{m+i}(d_i\, d_{2m-i}\,\omega_c^{2m} - a_i\, a_{2m-i})]$$

$$\left.\begin{array}{ll} k = 0, & \ell = 2m \text{ for } 2m \leqq n \\ k = 2m - n, & \ell = n \quad \text{for } 2m > n \end{array}\right\} \quad m = 1, 2, \dots, n \tag{8}$$

The proof of the lemma is given in the Appendix.

When the closed-loop characteristic polynomial $\phi_c(s)$ is replaced by Eq. (7), the weighting coefficients providing the Kessler pole location are obtained as

$$q_{n-m+1}/r = \sum_{i=k}^{\ell} (-1)^{m+i}\,[2^{-i(i-1)/2} \times 2^{-(2m-i)(2m-1-i)/2}\omega_c^{2m} - a_i\, a_{2m-i}]$$

$$\left.\begin{array}{ll} k = 0, & \ell = 2m \text{ for } 2m \leqq n \\ k = 2m - n, & \ell = n \quad \text{for } 2m > n \end{array}\right\} \quad m = 1, 2, \dots, n \tag{9}$$

For example, when the plant transfer function is given by $G(s) = 1/s^n$, the weighting coefficients of Eq. (9) are as shown in Table 2. Note that the weights q_2 and q_n are zero. The weighting coefficients of Eq. (9) may become negative, depending on the coefficients a_i of the open-loop characteristic polynomial. In that case, the matrix Q is no longer positive semidefinite. This is illustrated in the next example.

Example 1. Consider a second-order system given by

$$G(s) = \frac{1}{s\,(s + a)} \tag{10}$$

The weighting coefficients of the Kessler type are from Eq. (9)

$$q_1/r = (\omega_c^2/2)^2 > 0, \qquad q_2/r = -a^2 < 0 \tag{11}$$

The coefficient q_2 is negative irrespective of the sign of a.

Uniqueness of the Solution

Generally, an ARE has a unique positive definite solution if the system is controllable and the weighting matrix Q is posi-

Table 1 Root location of the Kessler polynomials ($\omega_c = 1$)

Order n	Roots	θ,[a] deg	Damping ratio ζ
1	-1	0	1.0
2	$-0.5(1 \pm j)$	45	0.707
3	-0.50, $-(0.25 \pm j0.43)$	60	0.5
4	$-0.25(1 \pm j)$ (multiple roots)	45	0.707
5	-0.25, $-(0.095 \pm j0.11)$, $-(0.28 \pm j0.33)$	49.5	0.649

[a]θ = maximum angle from the negative real axis.

Table 2 Weights for the plant $G(s) = 1/s^n$

Order	q_1/r	q_2/r	q_3/r	q_4/r	q_5/r
1	ω_c^2	—	—	—	—
2	$2^{-2}\omega_c^4$	0	—	—	—
3	$2^{-6}\omega_c^6$	0	0	—	—
4	$2^{-12}\omega_c^8$	0	$2^{-5}\omega_c^4$	0	—
5	$2^{-20}\omega_c^{10}$	0	$2^{-9}\omega_c^6$	$2^{-5}\omega_c^4$	0

tive definite. The problem is then whether the Kessler polynomial gives a unique optimal solution, because the weighting matrix Q is no longer restricted to being sign definite. This problem was solved by Molinari,[7] as shown in Lemma 2.

Lemma 2. Consider the LQR problem of Eqs. (1) and (2) normalized by an input weight. The Hamilton matrix is defined as

$$M = \begin{pmatrix} A & BB^T \\ Q & -A^T \end{pmatrix}$$

Assume that the pair (A, B) is controllable. Then, the following conditions are equivalent:

1) An ARE $PA + A^TP - PBB^TP + Q = 0$ has a unique real symmetric solution P satisfying $\mathrm{Re}\,\lambda\,(A - BB^TP) < 0$.

2) $\mathrm{Re}\,\lambda\,(M) \neq 0$, for all eigenvalues λ of M.

It is to be noted in the lemma that no sign definite property is assigned to Q. The characteristic equation of the Hamilton matrix in the present problem is equal to $\phi_c(s)\,\phi_c(-s)$, where $\phi_c(s)$ is the Kessler characteristic polynomial. As a result, property 2 in the Lemma 2 is always satisfied. Therefore, an ARE has a unique solution from property 1.

Control Property Given by Negative Weights

Circle Condition

For the LQR problems, the Kalman equation derived from the ARE is[1]

$$[I + K\Phi(-s)B]^T R [I + K\Phi(s)B]$$
$$= R + B^T\Phi^T(-s)Q\Phi(s)B \tag{12}$$

where $\Phi(s) = (sI - A)^{-1}$ and R is an input weighting matrix. If Q is a positive semidefinite matrix, the circle condition with $s = j\omega$ is satisfied for all ω such that

$$[I + K\Phi(-j\omega)B]^T R [I + K\Phi(j\omega)B] \geqq R \tag{13}$$

The LQR controllers have a low-sensitivity for all frequencies because the matrix $I + K\,\Phi\,(j\omega)B$ in Eq. (13) is the inverse of the sensitivity matrix. In particular, the gain margin of the loop transfer function $K\Phi(s)B$ of a single-input system is infinite and the phase margin is greater than 60 deg.

The circle condition does not hold for the Kessler pole location, because of the presence of some negative diagonal elements in the weighting matrix. This is shown in Example 1. For an LQR controller with a positive semidefinite matrix Q, it is not always necessary to satisfy the circle condition for the suppression of high-frequency noise. It is desirable, however, that every feedback system satisfies the condition up to or near the open-loop crossover frequency. This will be substantiated in the design examples in the application section.

Rolloff Property

For conventional single-input LQR problems, the relative degree of the loop transfer function $K\,\Phi\,(s)\,B \leq 1$, due to the circle condition and the positive definite property of the ARE solution. This means that the Nyquist plot asymptotically approaches the origin along the negative imaginary axis as $\omega \to \infty$, which in turn shows that the rolloff property is -20 dB/dec. If control laws are not of full state feedback, then a rolloff property of more than -20 dB/dec can be obtained.

The closed-loop characteristic polynomial $\phi_c(s)$ of the plant shown in Fig. 1 is given by

$$s^n + (a_1 + k_n)s^{n-1} + \cdots + (a_n + k_1) = 0 \tag{14}$$

From Eqs. (5) and (14), the feedback gain is given by

$$k_{n-i+1} = d_i\omega_c^i - a_i, \qquad i = 1, 2, \ldots, n \tag{15}$$

If it is desirable to put $k_i = 0$, from Eq. (15),

$$d_m = a_m\omega_c^{-m} \quad \text{for } m = n + 1 - i, \qquad i = 1, 2, \ldots, n \tag{16}$$

In particular, $d_1 = a_1/\omega_c$ for $k_n = 0$, which corresponds to a rolloff property of -40 dB/dec.

Example 2. Consider the same system as in Example 1 $(a > 0)$. To obtain $k_2 = 0$, $d_1 = a/\omega_c$ from Eq. (16). Then, the weighting coefficients are, from Eq. (8),

$$q_1/r = (d_2\omega_c^2)^2, \qquad q_2/r = -2d_2\omega_c^2 \tag{17}$$

where q_2 results in a negative weight. If the Kessler pole location is used, $d_1 = 1$ and $d_2 = ½$, and Eq. (17) corresponds to Eq. (11). In this case, however, the crossover frequency is restricted to $\omega_c = a$, and the system may not generally show desirable closed-loop characteristics. This problem will be examined in the next section as well.

Phase Lead Problem

Conventional LQR controllers that use all state feedbacks provide a maximum number of zeros for the open-loop system and, thus, maintain the least phase delay. Therefore, as shown in the next example, a phase lead not necessary for a classical controller is added. This can be avoided using negative weights.

Example 3. Consider the same second-order system as in Example 1, where it is assumed $a = 1$. When the optimal control is used, the gains shown in Fig. 1 are given as

$$k_1 = (q_1/r)^{½} \tag{18a}$$

$$k_2 = \{1 + q_2/r + 2(q_1/r)^{½}\}^{½} - 1 \tag{18b}$$

When $k_1 = 0.5$ is given, the classical choice of the gains is $k_2 = 0$. However, $k_2 = \sqrt{2} - 1$ for $q_2 = 0$ from Eqs. (18), and a useless phase lead is added. Hence, $q_2/r = -1$ has to be chosen in order to obtain $k_2 = 0$, where the positivity of the weights does not hold. On the other hand, this selection of weights corresponds to the weights of Eq. (11), which provide the Kessler pole location. Furthermore, when it is selected to be $\omega_c = 1$, this system corresponds to the one in Example 2, with the rolloff property of -40 dB/dec.

Application to Roll Autopilot Systems

The results derived in the previous sections are applied to a roll autopilot design for missiles to demonstrate the validity of the proposed method. The controlled models are shown in Fig. 2, and the nomenclature and the nominal values of the parameters are shown in Table 3.[5] The performance index is given as

$$J = \frac{1}{2}\int_0^\infty \left[\left(\frac{\delta_{CMX}}{\phi_{MX}}\right)^2 \phi^2 + \left(\frac{\delta_{CMX}}{\dot\phi_{MX}}\right)^2 \dot\phi^2 + \delta_C^2\right] \mathrm{d}t \tag{19}$$

Though the purpose is to improve the roll time constant, it is important for the design to take into account the effects of neglected high-frequency dynamics. For this purpose, the condition of stability robustness is also discussed.

Lemma 3.[8] Let the transfer function $G'(j\omega)$, which takes a multiplicative uncertainty $L(j\omega)$ of the nominal model

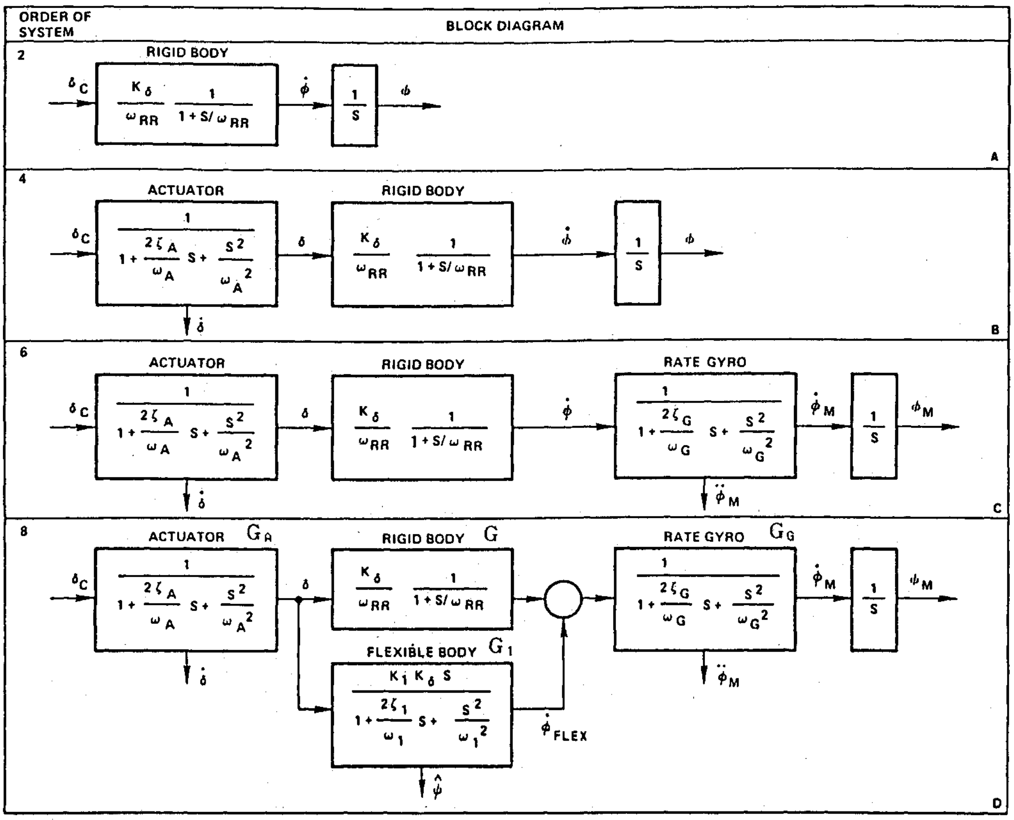

Fig. 2 Block diagrams of missile roll dynamics.[5]

$G(j\omega)$ into account, be given as

$$G'(j\omega)=[I+L(j\omega)]G(j\omega) \tag{20}$$

$$\bar{\sigma}[L(j\omega)]<\ell_m(j\omega)$$

where $\bar{\sigma}\,[L]$ denotes the maximum singular value of L. Then, the robust stability in the high-frequency region is guaranteed, if the nominal closed-loop system is asymptotically stable, and the following condition is satisfied

$$\bar{\sigma}\{KG(j\omega)[I+KG(j\omega)]^{-1}\}\cong\bar{\sigma}[KG(j\omega)] < 1/\ell_m(\omega) \tag{21}$$

For the system shown in Fig. 2, the values of $\ell_m(\omega)$ at the input break point are shown in Fig. 3. The curves ℓ_m and $1/\ell_m$ indicated by ① show the case where the actuator and gyro dynamics are ignored. The curve ② is the case where the first torsion mode is neglected. The boundary line of the robust stability is presented with an oblique line in Fig. 3.

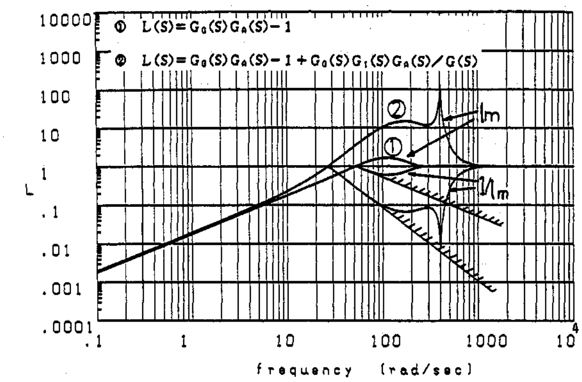

Fig. 3 Uncertainty boundaries.

Conventional Linear Quadratic Regulator

Nesline and Zarchan[5] designed LQR controllers for the systems shown in Fig. 2, where the nominal values of Table 3 are used to find the weights for their first design. They showed that the controllers have high crossover frequencies (HCF) and that their stabilities are not robust. As an example, the open-loop frequency response of the second-order system in Fig. 2a is shown as the HCF curve in Fig. 4. This controller makes the four-state system with the actuator dynamics (Fig. 2b) unstable. In the same way, the controller designed for the four-state system makes the six-state system with sensor dynamics (Fig. 2c) unstable. These results can be expected because the HCF curve in Fig. 4 does not satisfy the boundary line of robust stability for the unmodeled actuator and sensor dynamics. The conclusions they derived are, therefore, as follows.

Table 3 Definition of parameters and their nominal values[5]

Name	Definition	Value
ω_{RR}	Roll rate bandwidth	2 rad/s
K_δ	Fin effectiveness	9000 $1/s^2$
ω_A	Actuator bandwidth	100 rad/s
ζ_A	Actuator damping	0.65
ω_G	Rate gyro bandwidth	200 rad/s
ζ_G	Rate gyro damping	0.5
ω_1	Torsional mode frequency	400 rad/s
ζ_1	Torsional mode damping	0.01
K_1	Torsional mode gain	−0.001
ϕ_{MX}	Maximum desired roll angle	10 deg
$\dot{\phi}_{MX}$	Maximum desired roll rate	300 deg/s
δ_{CMX}	Maximum desired fin deflection	30 deg

1) The gain of the open-loop transfer function from the LQR method is constant regardless of the plant order, and is a function only of the weights.

2) A classical rule of thumb is to choose a crossover frequency that is about one-third of the actuator bandwidth.

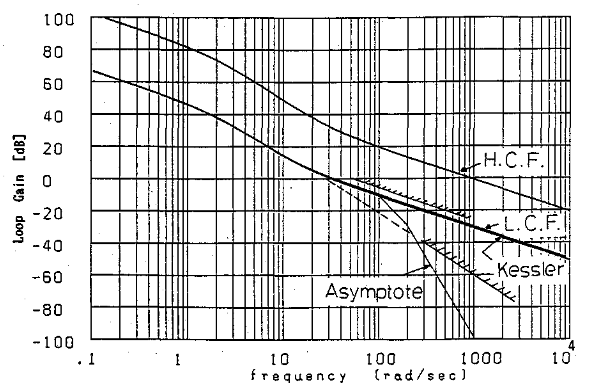

Fig. 4 Loop transfer functions for the second-order system.

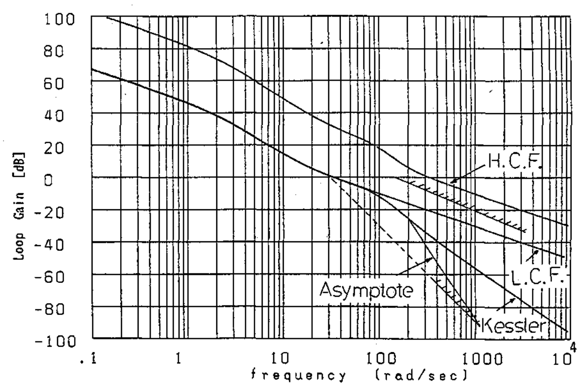

Fig. 6 Loop transfer functions for the fourth-order system.

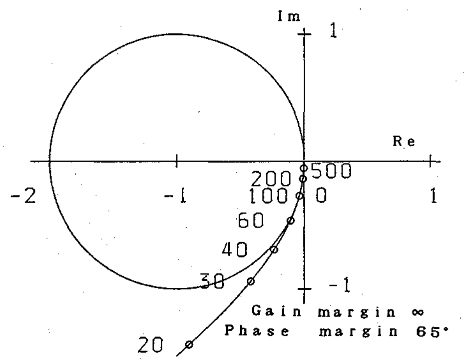

Fig. 5 Nyquist diagram with the Kessler-type controller (second-order system, $\omega_c = 30$ rad/s).

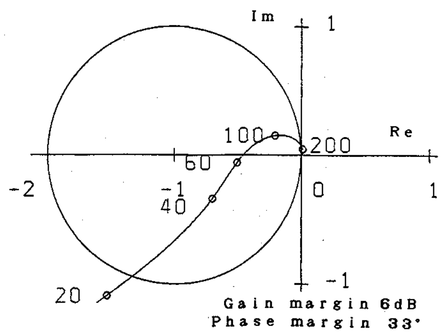

Fig. 7 Nyquist diagram with the Kessler-type controller (fourth-order system, $\omega_c = 132$ rad/s).

3) Modern control with low crossover frequency assigns open-loop zeros to the locations where the plant poles are canceled. As a result, a higher-order compensator is not required.

According to these observations, the new value of δ_{CMX} is selected to be 0.5 deg, resulting in the the low crossover frequency (LCF) system, with the desirable response shown in Fig. 4.

Design of Kessler System (Second-Order)

For the second-order system in Fig. 2a, a magnitude plot with the Kessler weights of $\omega_c = 30$ rad/s is also shown in Fig. 4. Although a negative weight is used in this design, the plot almost coincides with the LCF curve, and has desirable characteristics. This is because both pole patterns are equal in second-order systems. The Nyquist plot of the open-loop transfer function with the Kessler pole location is shown in Fig. 5. The plot enters the unit circle at 226 rad/s because the circle condition does not hold, due to the negative weight. This, however, is indistinguishable in Fig. 5, and may cause no problem in practice.

Design of Kessler System (Fourth-Order)

The controlled plant shown in Fig. 2b is of the four-state system, including actuator dynamics. Generally, actuators need to have good damping characteristics, and their bandwidths need to be much higher than the bandwidths of the units they control. In this example, $2\zeta_A\omega_A = 130$ rad/s, and improvements to the roll rate bandwidth ω_{RR} and the rolloff property given in the previous section are obtained simultaneously. That is, defining $\omega_c = \omega_{RR} + 2\zeta_A\omega_A = 132$ rad/s according to Eq. (16) results in a rolloff of -40 dB/dec.

The magnitude plots of the HCF, LCF, and Kessler pole location with a crossover frequency of 132 rad/s are shown in Fig. 6. The upper boundary of the robust stability is derived by ignoring the rate gyro dynamics, and the lower boundary by ignoring the gyro dynamics and the torsional mode. The magnitude plot of the HCF shows small variations near 100 and 200 rad/s, whereas the LCF curve is equal to the plot of the two-state plant in Fig. 4. (This illustrates property 3 in the previous subsection.) On the other hand, the Kessler pole location attains a rolloff of -40 dB/dec in the high-frequency region. The low-frequency portion is identical with the LCF plot, with a crossover frequency of 34 rad/s. It is to be noted that the frequency is automatically determined from Eq. (16).

A negative weight is used in this design, as in the two-state design, and the circle condition does not hold. The Nyquist plot is shown in Fig. 7, where the plot enters the unit circle at $\omega = 22$ rad/s and leaves at 170 rad/s. Since the phase lag becomes 360 deg for $\omega \to \infty$, the Nyquist plot approaches the origin along the positive real axis. Although the circle condition is not satisfied, the gain and phase margins are 6 dB and 33 deg, respectively. The return difference > 1 up to or near the open-loop crossover frequency.

Finally, the Kessler and the LCF plots in Figs. 4 and 6 do not satisfy the robust boundary determined by neglecting the first torsional mode. It is to be noted that the condition of robust stability using singular values results in a conservative solution, because the information pertaining to the phase is ignored. However, because of the rolloff property of -40 dB/dec in this example, the Kessler curve results in greater robust stability and attenuation of high-frequency noise than does the LCF curve.

Conclusion

This paper presents a method for selecting negative diagonal elements of the weighting matrix in a linear quadratic regulator (LQR) problem. The proposed method is based on the use of a polynomial as a desirable pole specification, and the weights are given in an analytical form. As a result, the damping and rolloff properties of LQR are improved, and the undesirable phase lead is avoided. These desirable characteristics are attained at the sacrifice of the circle condition.

Appendix: Proof of Lemma 1

Consider the following equations:

$$f(s) = \sum_{i=0}^{n} \alpha_i \omega_c^i s^{n-i}$$

$$F(s) = \sum_{i=0}^{2n} F_i s^i = f(s) f(-s) \tag{A1}$$

The $(2n-m)$th-order coefficient F_{2n-m} of $F(s)$ is given by

$$F_{2n-m} = \sum_{i=0}^{m} (-1)^{n-m+i} \alpha_i \alpha_{m-i} \omega_c^m \quad \text{for} \quad m = 0, 1, \ldots, n \tag{A2}$$

$$F_{2n-m} = \sum_{i=m-n}^{n} (-1)^{n-m+i} \alpha_i \alpha_{m-i} \omega_c^m \quad \text{for} \quad m = n+1, n+2, \ldots, 2n \tag{A3}$$

From Eqs. (A2) and (A3), F is given by

$$F(s) = \sum_{m=0}^{2n} \left\{ \sum_{i=k}^{\ell} [(-1)^{n-m+i} \alpha_i \alpha_{m-i}] \omega_c^m s^{2n-m} \right\}$$

$$k = 0, \qquad \ell = m \quad \text{for } m \leqq n$$

$$k = m-n, \qquad \ell = n \quad \text{for } m > n$$

Because the odd-order terms of $f(s) f(-s)$ are cancelled, $F(s)$ is rewritten as

$$F(s) = \sum_{m=0}^{n} \left\{ \sum_{i=k}^{\ell} [(-1)^{n+i} \alpha_i \alpha_{2m-i}] \omega_c^{2m} s^{2(n-m)} \right\} \tag{A4}$$

where k and ℓ are defined in Eq. (8). From Eqs. (A4) and (5), the following equations are obtained.

$$\phi_c(s)\phi_c(-s) = \sum_{m=0}^{n} \left\{ \sum_{i=k}^{\ell} [(-1)^{n+i} d_i d_{2m-i}] \omega_c^{2m} s^{2(n-m)} \right\} \tag{A5a}$$

$$\phi(s)\phi(-s) = \sum_{m=0}^{n} \left\{ \sum_{i=k}^{\ell} [(-1)^{n+i} a_i a_{2m-i}] s^{2(n-m)} \right\} \tag{A5b}$$

$$(1/r)\phi_0(-s) Q \phi_0(s) = \sum_{m=1}^{n} [(-1)^{n-m} \times (q_{n-m+1}/r) s^{2(n-m)}] / \phi(s)\phi(-s) \tag{A5c}$$

Equation (8) is derived by substituting Eq. (A5) for Eq. (5).

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References

[1]Kwakernaak, H., and Sivan, R., *Linear Optimal Control Systems*, Wiley Interscience, New York, 1972, Chap. 3.

[2]Johnson, M. A., and Grimble, M. J., "Recent Trends in Linear Optimal Quadratic Multivariable Control System Design," *IEEE Proceedings*, Vol. 134, Pt. D, No. 1, Jan. 1987, pp. 53–71.

[3]Hayase, M., "Linear Constant Coefficient Optimum Control Problems in which the Weighting Matrices Are Not Limited to Be Positive Semi-Definite," *Transactions of the Society of Instrument and Control Engineerings* (in Japanese), Vol. 9, No. 3, June 1973, pp. 311–318.

[4]Shin, V., and Chen, C., "On the Weighting Factors of the Quadratic Criterion in Optimal Control," *International Journal of Control*, Vol. 19, May 1974, pp. 947–955.

[5]Nesline, F. W., and Zarchan, P., "Why Modern Control Can Go Unstable in Practice," *Journal of Guidance, Control, and Dynamics*, Vol. 7, No. 4, 1984, pp. 495–500.

[6]Kalman, R. E., "When Is a Linear Control System Optimal," *Transactions of American Society of Mechanical Engineers*, *Journal of Basic Engineering*, Vol. 86D, March 1964, pp. 51–60.

[7]Molinari, B. P., "The Stable Regulator Problem and Its Inverse," *IEEE Transactions on Automatic Control*, Vol. AC-18, No. 5, Oct. 1973, pp. 454–459.

[8]Doyle, J. C., and Stein, G., "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis," *IEEE Transactions on Automatic Control*, Vol. AC-26, No. 1, Feb. 1981, pp. 4–16.